

Bijection between spin $S = \frac{p^M-1}{2}$ and a cluster of M spins $\sigma = \frac{p-1}{2}$

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Abstract

We propose a general method by which a spin- S is decomposed into spins less than S . We have obtain the exact mapping between spin $S = \frac{p^M-1}{2}$ and a cluster of M spins $\sigma = \frac{p-1}{2}$. We have discuss the possible applications of such transformations. In particular we have show how a general $d+1$ dimensional spin- $\frac{p-1}{2}$ model with general interactions can be reduced to d -dimensional spin- S model with $S = \frac{p^M-1}{2}$.

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I. INTRODUCTION

The investigation of spin- S models is important for applications and also for clarification of critical phenomena. The spin-1/2 Ising model has been extensively investigated because it has very wide applications to many interesting problems in different scientific areas. The general spin-1 model with up-down symmetry was introduced by Blume, Emery and Griffits (BEG) [1] for a model of $He^3 - He^4$ mixtures. A spin-3/2 model was proposed by Krinsky and Mukamel [2] for a model of ternary fluid mixtures. The spin- S model with higher spin ($S > 3/2$) have been studied in much less detail because of computational complexity increasing with spin.

So far a few exact solution have been obtained for spin- S models. In 1944 the spin-1/2 square lattice Ising model has been solved exactly by Onsager in his seminal paper [3]. All of the numerous attempts to extend the exact solution to the systems with $S > 1/2$ are failed, except for the exact results obtained for some high-spin models than can be reduced to known solvable models, like free-fermion models [4–6], spin-1/2 Ising model [7–9], Ashkin-Teller model [10, 11] under certain constrained conditions, which imposed on different coupling constants in those spin- S models. Mapping between models is an important tool for the study of exactly solvable models. The aim of this paper is to present one of such mapping, namely, the bijection (one-to-one transformation) between a general spin- S systems, with $S = \frac{p^M - 1}{2}$ and a system consist from a cluster of M spins σ with $\sigma = \frac{p-1}{2}$.

The present article is organized as follows. In section II we present the general spin-spin transformations between spin $S = \frac{2^M - 1}{2}$ and a cluster of M spins $\sigma = \frac{p-1}{2}$. In section III we give the expression for the general inverse spin-spin transformation. In the next section we discuss the application of our funding. In particular we have show that a general $d + 1$ dimensional spin- $\frac{p-1}{2}$ model with general interactions can be reduced to d-dimensional spin- S model with $S = \frac{p^M - 1}{2}$ and finally in section V we give our conclusions.

II. A GENERAL SPIN-SPIN TRANSFORMATION

Consider a system of N particles with spin variable $S = (p^M - 1)/2$ with p and M are a positive integer. For the j ($j = 1, 2, \dots, N$) particle, S_j is the z component of the spin operator and its eigenvalues are $\{S, S - 1, \dots, -S + 1, -S\}$. The system has a Hamiltonian

$H(\{S_j\}) \equiv H(S_1, S_2, \dots, S_N)$ and the partition function given by

$$Z = \sum_{\{S_j\}} \exp [-\beta H(\{S_j\})], \quad (1)$$

where the sum is over all $(2S + 1)^N$ possible spin configurations.

Let us consider the following representation for the eigenvalues of the z component of the spin operator S_j

$$S_j = \sum_{i=1}^M p^{i-1} \sigma_{i,j} = \sigma_{1,j} + p \sigma_{2,j} + \dots + p^{M-1} \sigma_{M,j}, \quad (2)$$

where $\sigma_{i,j}$ are variables which takes values $\{\frac{p-1}{2}, \frac{p-3}{2}, \dots, -\frac{p-1}{2}\}$. For even $p = 2k$ we have a clusters of M half integer spins $\sigma = k - 1/2$ which represent the half integer spin $S = \frac{(2k)^M - 1}{2}$ and for odd $p = 2k + 1$ we have a clusters of M integer spins $\sigma = k$ which represent the integer spin $S = \frac{(2k+1)^M - 1}{2}$.

The transformation given by Eq. (2) gives a bijection (one-to-one correspondence) between spin $S = (p^M - 1)/2$ and a cluster of M spins $\{\sigma_1, \sigma_2, \dots, \sigma_M\}$ where $\sigma_i = (p - 1)/2$ for $i = 1, 2, \dots, M$. The total number of spin components of spin $S = (p^M - 1)/2$ are

$$2S + 1 = p^M,$$

which is exactly the total number of spin configurations of a cluster of M spins $\sigma = (p - 1)/2$. Thus we can say that the set of spin- S and set of spins $\{\sigma_1, \sigma_2, \dots, \sigma_M\}$ are equivalent. Note that in the case $p = 2$, we obtained the condition

$$2S + 1 = 2^M,$$

which are necessary condition to express the operator of spin- S in terms of fermions [12].

By means of Eq. (2) the Hamiltonian $H(\{S_j\})$ may be expressed as a function of the $\sigma_{i,j}$

$$H(\{S_j\}) = H(\{S_j(\{\sigma_{i,j}\})\}).$$

We can therefore say that the new spin variables σ_i are independent and consequently the summation in $\sum_{\sigma_1, \sigma_2, \dots, \sigma_M} \exp (-\beta H(\{\sigma_i\}))$ can be carried out independently. So we may write the partition function as the sum over all possible (p^{MN}) spin configurations of a cluster of M spins $\sigma_i = \frac{p-1}{2}$ ($i = 1, 2, \dots, M$) instead of taking sum over $2S + 1$ spin configuration of spin- S

$$Z = \sum_{\{S_j\}} \exp [-\beta H(\{S_j\})] = \sum_{\{\sigma_{i,j}\}} \exp [-\beta H(\{S_j(\{\sigma_{i,j}\})\})] \quad (3)$$

The situation is different from the case considered by Griffiths in his paper [13], where he introduced the mapping between spin $S = p/2$ and a cluster of p spins $\sigma = 1/2$ in the following form

$$S = \sum_{i=1}^p \sigma_i = \sigma_1 + \sigma_2 + \dots + \sigma_p. \quad (4)$$

In that case the mapping is not a bijection and the Eq. (3) is changed to

$$\sum_{\{S_j\}} \exp(-\beta H(\{S_j\})) = \sum_{\{\sigma_{i,j}\}} \exp(-\beta \bar{H}(\{\sigma_{i,j}\})), \quad (5)$$

where Griffiths introduced the so called weight function $W_j(\sigma_{1,j}, \sigma_{2,j}, \dots, \sigma_{p,j})$ in definition of the analog Hamiltonian $\bar{H}(\{\sigma_{i,j}\})$

$$\bar{H}(\{\sigma_{i,j}\}) = H(S(\{\sigma_{i,j}\})) - \frac{1}{\beta} \sum_{j=1}^N \ln W_j(\{\sigma_{i,j}\}) \quad (6)$$

where $\beta = (kT)^{-1}$ is the inverse temperature and analog Hamiltonian is temperature dependent. In our case the weight function is equal to 1 ($W_j = 1$) and Eqs. (5) and (6) become Eq. (3).

III. A GENERAL INVERSE SPIN-SPIN TRANSFORMATION

Let us now consider the spin-spin transformation inverse to transformation given by Eq. (2). We want to express a spin $\sigma = \frac{p-1}{2}$ as a function of spin- S . The general expression of such inverse transformation can be written in the following form

$$\sigma_m(s) = \sum_{j=0}^{2S} A_j(s) P_{j,m}, \quad (7)$$

where

$$A_j(s) = \frac{(-1)^{2S+j}}{j!(2S-j)!} \prod_{\substack{i=0 \\ i \neq j}}^{2S} (s + S - i), \quad (8)$$

$P_{j,m}$ are the projection of spin- σ and $m = 1, 2, \dots, M'$ with $M' > M$, where M' represents the number of permutation of the $\{P_{1,m}, \dots, P_{p^M,m}\}$. Let us consider as an first example the case $p = 2$, so the only possible values of $P_{j,m}$ is $\pm \frac{1}{2}$ (in reference [14] the $P_{j,m}$ was considered as ± 1). For the case of spin-1 we have $p = 3$, the possible values of $P_{j,m}$ should be $\{-1, 0, 1\}$. Similarly for $p = 4$, the only possible values $P_{j,m}$ are $\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. In general the possible

values of $P_{j,m}$ are $\{-S, -S + 1, \dots, S - 1, S\}$. It is clear that the transformation given by Eqs. (7) and (8) is the most general transformation which maps the set of spins $\sigma_m = \frac{p-1}{2}$ ($m = 1, 2, \dots, M'$) with arbitrary integer values of p to a general spin- S , where S can take as integer as half-integer values.

It is also clear from Eqs. (7) and (8) that $\sigma_m(s)$ is a polynomial in s of degree $2S$, which we can write in the following form

$$\sigma_m(s) = \sum_{j=0}^{2S} \alpha_{m,j} s^j, \quad (9)$$

whose coefficient $\alpha_{m,j}$ can be defined using the result obtained in reference [6].

Let us chose the special values of $P_{j,m}$ that we can map exactly M particles with spin $\sigma = (p-1)/2$ to spin $S = (p^M - 1)/2$

$$\sigma_m(s) = \sum_{j=0}^{2S} \frac{(-1)^{2S+j} P_{j,m}}{j!(2S-j)!} \prod_{\substack{i=0 \\ i \neq j}}^{2S} (s + S - i), \quad (10)$$

where m take integer values from 1 to M and $P_{j,m}$ are given by

$$P_{j,m} = [j p^{m-M}] - p [j p^{m-1-M}] - \frac{p-1}{2}, \quad (11)$$

by $[x]$ we mean the less integer of any real x . With such values of $P_{j,m}$ the Eq. (7) and (8) gives us exactly M different values of σ_m ($m = 1, 2, \dots, M$) and such transformation again give us one-to-one correspondence between set of spins σ_m ($m = 1, 2, \dots, M$) and half-integer spin- S with $S = \frac{p^M-1}{2}$. Note, that this special values of $P_{j,m}$ already was obtained for the particular case $p = 2$ in reference [14]. It is also worth to note that the transformation given by Eq. (7) and (8) also is a generalization of the previous result obtained by Joseph [15] for the case $p = 2$ and for the particular case of the projection $P_{j,m}$ ($P_{j,m} = 1$ for all values of j and m).

IV. APPLICATIONS

In Sec. II we consider a system of N particles with spin variable S_j ($j = 1, 2, \dots, N$) and represent the spin variables for the j -th particle, S_j in the form Eq. (2), which give us a bijection between spin $S = (p^M - 1)/2$ and a cluster of M spins $(\sigma_1, \sigma_2, \dots, \sigma_M)$ where $\sigma_i = (p-1)/2$ for $i = 1, 2, \dots, M$. In Sec. III we consider the spin-spin transformation inverse

to transformation given by Eq. (2). From the general inverse spin-spin transformation (see Eqs. (7) and (8)) we chose the special values of $P_{j,m}$ and obtain the inverse spin-spin transformations given by Eqs. (10) and (11) that can map exactly M particles with spin $\sigma = (p-1)/2$ to spin $S = (p^M - 1)/2$ inverse to transformation system of N particles with spin variable S_j ($j = 1, 2, \dots, N$).

Let us now consider few examples:

A. Spin-spin transformation for $M = 2$ and $p = 2$

For the case $M = 2$ and $p = 2$ the transformations given by Eqs. (2), (10) and (11) reads as

$$S_j = \sigma_{1,j} + 2\sigma_{2,j} \quad (12)$$

and $\sigma_{1,j}$ and $\sigma_{2,j}$ are given by

$$\sigma_{1,j} = \frac{13}{12}s - \frac{1}{3}s^3, \quad (13)$$

$$\sigma_{2,j} = -\frac{7}{6}s + \frac{2}{3}s^3. \quad (14)$$

The above transformation gives us bijection between spin $S = 3/2$ and pair of Ising spins $\sigma = 1/2$. Such transformation has been already used by many authors [4, 6, 11, 16]. For example in the paper [4] the most general spin 3/2 model with up-down symmetry was solved exactly along two lines in the parameter space of the model with the help of transformation given by Eqs. (12) and (14). In the paper [16] the authors used the transformation given by Eqs. (12) and (14) to show that spin-3/2 model is equivalent to the two-layer Ising model with the spin-1/2. In paper [11] the new type of exact solution to the generalized Ashkin-Teller model was found with the help of the above mentioned transformation and in the paper [6] the authors present a set of exactly solvable models, with half-integer spin- S on a square-type lattice including the case of the spin-3/2.

B. Spin-spin transformation for $M = 3$ and $p = 2$

For the case $M = 3$ and $p = 2$ the transformations given by Eqs. (2), (10) and (11) reads as

$$S_j = \sigma_{1,j} + 2\sigma_{2,j} + 4\sigma_{3,j} \quad (15)$$

and $\sigma_{1,j}$, $\sigma_{2,j}$ and $\sigma_{3,j}$ are given by

$$\sigma_{1,j} = \frac{1}{252}s^7 - \frac{61}{720}s^5 + \frac{301}{576}s^3 - \frac{30251}{26880}s, \quad (16)$$

$$\sigma_{2,j} = -\frac{1}{630}s^7 + \frac{17}{360}s^5 - \frac{637}{1440}s^3 + \frac{14887}{13440}s, \quad (17)$$

$$\sigma_{3,j} = -\frac{4}{315}s^7 + \frac{11}{45}s^5 - \frac{217}{180}s^3 + \frac{2161}{1680}s. \quad (18)$$

The above transformation gives us bijection between spin $S = 7/2$ and three Ising spins $\sigma = 1/2$. Such transformation are new and the possible applications of that transformation can be described as follows:

1. One can used such transformation to show that a d-dimensional general spin - 7/2 model is equivalent to the three d-dimensional layer of spin-1/2 model.
2. Based on such correspondence one can try to find exact solvable cases for a general two-dimensional spin-7/2 model.

Let us consider for example a most general spin-7/2 model with up-down symmetry on a d-dimensional lattice G , whose Hamiltonian is given by

$$-\beta H(S) = \sum_{\langle i,j \rangle} \left\{ \sum_{\substack{\alpha, \beta=1 \\ \beta \geq \alpha \\ \alpha+\beta=\text{even}}}^{2S} \frac{J_{\alpha, \beta}}{2} (S_i^\alpha S_j^\beta + S_i^\beta S_j^\alpha) + \sum_{a=1}^{\frac{2S-1}{2}} \frac{\gamma h_{2a}}{2} (S_i^{2a} + S_j^{2a}) \right\}, \quad (19)$$

with $S = 7/2$, where $\beta = 1/kT$ us usual and γ is coordination number of the lattice. Here, each spin variable S_i is defined at a lattice site and takes one of the following value $\{7/2, 5/2, \dots, -7/2\}$. The summation over α and β (here β cannot be confused with inverse temperature) from 1 to 7 and $\langle i, j \rangle$ indicates summation over the pairs of nearest neighbor sites. We have 16 nearest neighbor interactions terms with interaction constant $J_{\alpha, \beta}$ with $\alpha, \beta = 1, 2, \dots, 7$, $\beta \geq \alpha$ and $\alpha + \beta = \text{even}$. We have also three fields h_{2a} with $a = 1, 2, 3$. Totally we have 19 interactions constants $J_{\alpha, \beta}$ and h_{2a} . Hence the Hamiltonian (19) represents a wide class of systems. The free energy of the system f_S is defined by

$$-\beta f_S = \lim_{V \rightarrow \infty} \frac{1}{V} \ln Z, \quad (20)$$

where V is the volume of the system and Z is the partition function given by

$$Z = \sum_{\{S_i\}} \exp [-\beta H(S)]. \quad (21)$$

Now we apply the transformations given by Eqs. (15) to express S_i in terms of three Ising spins $\sigma_{1,i}, \sigma_{2,i}$ and $\sigma_{3,i}$. Then we have the following Hamiltonian from Eq. (19) up to a constant

$$\begin{aligned}
-\beta H(\sigma) = & \sum_{\langle i,j \rangle} \left\{ \sum_{a=1}^3 K_{a,a} \sigma_{a,i} \sigma_{a,j} + \sum_{\substack{a,b=1 \\ b>a}}^3 \frac{K_{a,b}}{2} (\sigma_{a,i} \sigma_{b,j} + \sigma_{b,i} \sigma_{a,j}) + \sum_{\substack{a,b=1 \\ b>a}}^3 \frac{K_{b,a}}{2} (\sigma_{a,i} \sigma_{b,i} + \sigma_{a,j} \sigma_{b,j}) \right. \\
& + \sum_{\substack{a,b,c=1 \\ c \neq b \neq a \\ c>b}}^3 \frac{R_{a,b,c}}{2} \sigma_{a,i} \sigma_{a,j} (\sigma_{b,i} \sigma_{c,j} + \sigma_{b,j} \sigma_{c,i}) + \sum_{\substack{a,b,c=1 \\ c \neq b \neq a \\ c>b}}^3 \frac{R_{a,c,b}}{2} \sigma_{a,i} \sigma_{a,j} (\sigma_{b,i} \sigma_{c,i} + \sigma_{b,j} \sigma_{c,j}) \\
& \left. + \sum_{\substack{a,b=1 \\ b>a}}^3 R_{a,b} \sigma_{a,i} \sigma_{a,j} \sigma_{b,i} \sigma_{b,j} + R \sigma_{1,i} \sigma_{2,i} \sigma_{3,i} \sigma_{1,j} \sigma_{2,j} \sigma_{3,j} \right\} \tag{22}
\end{aligned}$$

which can be considered as a Hamiltonian of the three layer of spin- σ ($\sigma = 1/2$) model, where each layer is represented by d-dimensional lattice G . We have 9 two-spin interactions given by interaction constants $K_{a,b}$ ($a, b = 1, 2, 3$), 9 four-spin interactions given by 6 interaction constants $R_{a,b,c}$ ($a \neq b \neq c$) and three interaction constants $R_{a,b}$ ($b > a$) and one six-spin interaction given by interaction constant R . Totally we again have 19 interaction constants ($R, R_{a,b}, R_{a,b,c}, K_{a,b}$) (as in the case of the spin-7/2 model). Dependence of the new coefficients $R, R_{a,b}, R_{a,b,c}$ from old ones $J_{\alpha,\beta}$ and h_a are given in Appendix (see Eqs. (67) - (85)). Thus we have established one-to-one correspondence between spin-7/2 model and three layers spin-1/2 model.

The above constructions can be easily extend to arbitrary spin- S (with $S = \frac{p^M-1}{2}$) model to show a one-to-one correspondence between d-dimensional spin- S model and M d-dimensional layers of spin- $\frac{p-1}{2}$ model.

One can consider some particular cases of the Hamiltonian given by Eq. (22). For example, if we imposed the following conditions

$$\begin{aligned}
R &= 0, \\
R_{a,b,c} &= 0 \quad \text{for} \quad a \neq b \neq c, \\
R_{a,b} &= 0 \quad \text{for} \quad b > a, \\
K_{a,b} &= 0 \quad \text{for} \quad b > a, \tag{23}
\end{aligned}$$

to cancel 6-spin interactions, four-spin interactions and some two-spin interactions we will

obtained from Eq. (22) the following Hamiltonian

$$-\beta H(\sigma) = \sum_{\langle ij \rangle} \sum_{a=1}^3 K_{a,a} \sigma_{a,i} \sigma_{a,j} + \sum_i \sum_{\substack{a,b=1 \\ b>a}}^3 \gamma K_{b,a} \sigma_{a,i} \sigma_{b,i}. \quad (24)$$

In this way, the system $H(S)$ is expressed in terms of three Ising model of σ_1, σ_2 and σ_3 , each on the lattice G coupled by glue interactions $K_{b,a}$, $b > a$.

If we imposed the additional to Eq. (23) conditions

$$K_{1,1} = K_{2,2} = K_{3,3} \quad \text{and} \quad K_{2,1} = K_{3,2} = K_{3,1} \quad (25)$$

we will arrive to the Ising model with periodic boundary conditions with following Hamiltonian

$$-\beta H(\sigma) = K_1 \sum_{\langle ij \rangle} (\sigma_{1,i} \sigma_{1,j} + \sigma_{2,i} \sigma_{2,j} + \sigma_{3,i} \sigma_{3,j}) + \gamma K_2 \sum_i (\sigma_{1,i} \sigma_{2,i} + \sigma_{2,i} \sigma_{3,i} + \sigma_{1,i} \sigma_{3,i}) \quad (26)$$

with $K_1 = \frac{105840}{19} J_{7,7}$ and $K_2 = 360 h_6$. The Hamiltonian given by Eq. (26) is equivalent to the spin - 7/2 Hamiltonian given by Eq. (19) with the following constraint on the coupling constants

$$J_{2,2} = J_{2,4} = J_{2,6} = J_{4,4} = J_{4,6} = J_{6,6} = 0, \quad (27)$$

$$J_{1,1} = \frac{8991341559}{389120} J_{7,7}, \quad J_{1,3} = -\frac{1424553921}{48640} J_{7,7}, \quad J_{1,5} = \frac{62601021}{12160} J_{7,7}, \quad (28)$$

$$J_{1,7} = -\frac{764019}{3040} J_{7,7}, \quad J_{3,3} = \frac{52080091}{4864} J_{7,7}, \quad J_{3,5} = -\frac{12187819}{3040} J_{7,7}, \quad (29)$$

$$J_{3,7} = \frac{30625}{152} J_{7,7}, \quad J_{5,5} = \frac{30821}{80} J_{7,7}, \quad J_{5,7} = -\frac{7441}{190} J_{7,7}, \quad (30)$$

and

$$h_2 = \frac{259}{16} h_6, \quad h_4 = -\frac{35}{4} h_6. \quad (31)$$

If we imposed another additional to Eq. (23) conditions

$$K_{1,1} = K_{2,2} = K_{3,3}, \quad K_{2,1} = K_{3,2} \quad \text{and} \quad K_{3,1} = 0, \quad (32)$$

we will arrive to the Ising model with free boundary conditions with following Hamiltonian

$$-\beta H(\sigma) = K_1 \sum_{\langle ij \rangle} (\sigma_{1,i} \sigma_{1,j} + \sigma_{2,i} \sigma_{2,j} + \sigma_{3,i} \sigma_{3,j}) + \gamma K_3 \sum_i (\sigma_{1,i} \sigma_{2,i} + \sigma_{2,i} \sigma_{3,i}), \quad (33)$$

with $K_1 = \frac{105840}{19} J_{7,7}$ and $K_3 = -90h_6$. The Hamiltonian given by Eq. (33) is equivalent to the spin-7/2 Hamiltonian given by Eq. (19) with the constraints given by Eqs. (27) - (30) and

$$h_2 = \frac{1429}{16} h_6, \quad h_4 = -20 h_6. \quad (34)$$

We can note that under conditions given by Eqs. (27) - (31) and (34) the spin- S Hamiltonian have two free parameters.

One can also obtain some exactly solvable case by putting the glue interactions $K_{b,a}$ to zero. In that case we have three non interacting Ising model, each on the d -dimensional lattice G , and in the case $d = 2$ we will obtain exact solvable case for spin - 7/2 model under following conditions between interactions constant $J_{\alpha,\beta}$ and h_a for original spin-7/2 model

$$J_{1,1} = \frac{6276855}{512} J_{5,7} - \frac{47629545}{1024} J_{7,7} + \frac{2557047}{1792} J_{5,5}, \quad (35)$$

$$J_{1,3} = -\frac{2064839}{256} J_{5,7} + \frac{14979545}{256} J_{7,7} - \frac{16765}{16} J_{5,5}, \quad (36)$$

$$J_{1,5} = -\frac{87805}{32} J_{5,7} - \frac{2794715}{32} J_{7,7} - \frac{311}{8} J_{5,5}, \quad (37)$$

$$J_{1,7} = \frac{3709}{16} J_{5,7} + \frac{97205}{16} J_{7,7} + \frac{50}{7} J_{5,5} \quad (38)$$

$$J_{3,3} = \frac{135485}{32} J_{5,7} + \frac{1225}{4} J_{5,5} + \frac{14984641}{256} J_{7,7}, \quad (39)$$

$$J_{3,5} = -\frac{3871}{16} J_{5,7} - 35 J_{5,5}, \quad (40)$$

$$J_{3,7} = -\frac{35}{2} J_{5,7} - \frac{3871}{8} J_{7,7}, \quad (41)$$

$$J_{2,2} = J_{2,4} = J_{2,6} = J_{4,4} = J_{4,6} = J_{6,6} = 0 \quad \text{and} \quad h_2 = h_4 = h_6 = 0. \quad (42)$$

The Hamiltonian under conditions given by Eqs. (35) - (42) is reduced to

$$-\beta H(\sigma) = \sum_{\langle i,j \rangle} (K_{1,1}\sigma_{1,i}\sigma_{1,j} + K_{2,2}\sigma_{2,i}\sigma_{2,j} + K_{3,3}\sigma_{3,i}\sigma_{3,j}) \quad (43)$$

where interaction constants $K_{1,1}$, $K_{2,2}$ and $K_{3,3}$ are given by

$$K_{1,1} = \frac{114975}{16} J_{5,7} + \frac{11432925}{64} J_{7,7} + \frac{1125}{4} J_{5,5},$$

$$K_{2,2} = 91350 J_{5,7} + \frac{7397775}{4} J_{7,7} + 4500 J_{5,5},$$

$$K_{3,3} = -88200 J_{5,7} - 2061675 J_{7,7} - 3600 J_{5,5}.$$

Thus we have show that three non interacting two-dimensional exactly solvable Ising model are equivalent to two-dimensional spin-7/2 model under conditions given by Eqs. (35) - (42).

C. Spin-spin transformation for $M = 2$ and $p = 3$

For the case $M = 2$ and $p = 3$ the transformations given by Eqs. (2), (10) and (11) reads as

$$S_j = \sigma_{1,j} + 3\sigma_{2,j} \quad (44)$$

and $\sigma_{1,j}$ and $\sigma_{2,j}$ are given by

$$\begin{aligned} \sigma_{1,j} &= s(s^2 - 1) \left(-\frac{27}{560}s^2 + \frac{1}{560}s^4 + \frac{139}{420} \right), \\ \sigma_{2,j} &= s(s^2 - 9) \left(\frac{57}{560}s^2 - \frac{3}{560}s^4 - \frac{31}{140} \right), \end{aligned} \quad (45)$$

The above transformations gives us bijection between spin $S = 4$ and pair of spins with $\sigma = 1$. Such transformation can be used for example to show that spin - 4 model is equivalent to the two-layer spin-1 model.

D. Spin-spin transformation for $M = 3$ and $p = 3$

For the case $M = 3$ and $p = 3$ the transformations given by Eqs. (2), (10) and (11) reads as

$$S_j = \sigma_{1,j} + 3\sigma_{2,j} + 9\sigma_{3,j} \quad (46)$$

and $\sigma_{1,j}$, $\sigma_{2,j}$ and $\sigma_{3,j}$ are given by

$$\begin{aligned} \sigma_{1,j} &= \left(\frac{15184387919}{1918955630592000}s^2 - \frac{66791923009387}{192008442802176000000}s^4 + \frac{22371900997}{2643031196467200000}s^6 - \frac{5057645209}{40579872915456000000}s^8 \right. \\ &\quad \left. - \frac{17330419}{226767340800} + \frac{2967383}{2608706115993600000}s^{10} - \frac{664843}{105721247858688000000}s^{12} + \frac{883}{45812540738764800000}s^{14} \right. \\ &\quad \left. - \frac{1}{39836991946752000000}s^{16} \right) s(s^2 - 1)(s^2 - 2^2)(s^2 - 3^2)(s^2 - 4^2), \end{aligned} \quad (47)$$

$$\begin{aligned} \sigma_{2,j} &= \left(-\frac{192181663909}{624923050752000000}s^2 + \frac{15276178774039}{427447366714368000000}s^4 - \frac{3162180475127}{1496065783500288000000}s^6 + \frac{88912189981}{1329836252000256000000}s^8 \right. \\ &\quad \left. + \frac{148211081}{128501493120000} - \frac{13780223389}{11968526268002304000000}s^{10} + \frac{126626341}{11968526268002304000000}s^{12} - \frac{193309}{3989508756000768000000}s^{14} \right. \\ &\quad \left. + \frac{173}{1994754378000384000000}s^{16} \right) s(s^2 - 1)(s^2 - 8^2)(s^2 - 9^2)(s^2 - 10^2), \end{aligned} \quad (48)$$

$$\begin{aligned} \sigma_{3,j} &= \left(\frac{841457709}{2345390215168000}s^2 - \frac{627741171441}{16417731506176000000}s^4 + \frac{25815639}{14857675571200000}s^6 - \frac{77436279}{1836948979712000000}s^8 \right. \\ &\quad \left. - \frac{15097}{23796572800} + \frac{7713}{13121064140800000}s^{10} - \frac{47463}{10103219388416000000}s^{12} + \frac{261}{13134185204940800000}s^{14} \right. \\ &\quad \left. - \frac{9}{262683704098816000000}s^{16} \right) s(s^2 - 3^2)(s^2 - 6^2)(s^2 - 9^2)(s^2 - 12^2). \end{aligned} \quad (49)$$

The above transformations gives us bijection between spin $S = 13$ and a cluster of three spins with $\sigma = 1$. Such transformation can be used for example to show that spin-13 model is equivalent to the three-layer spin-1 model.

E. A d -dimensional Ising model mapping onto single-particle spin- S model

In 70's Joseph [15], discussed the single-particle mapping onto Shrödinger exchange operators [19], using this approach Joseph also discussed the nature of the transition in non-linear spin- S Ising model for both cases: half-odd integer spin [20] and integer spin [21]. In this section we propose some similar approach to discuss the d -dimensional Ising model mapping onto single-particle spin- S model. Using the presented bijections between spin- S ($S = \frac{p^M-1}{2}$) and sets of M spins $(\sigma_1, \sigma_2, \dots, \sigma_M)$ with $\sigma_m = \frac{p-1}{2}$ is a mapping between a d -dimensional Ising model and a single-particle spin- S model.

Let us consider the general d -dimensional Ising model given by the following Hamiltonian.

$$-\beta H_M(s) = \sum_{\langle m, m' \rangle} J_{m, m'} \sigma_m(s) \sigma_{m'}(s) + h \sum_m \sigma_m(s), \quad (50)$$

where $J_{m, m'}$ is the coupling term between two spins $\sigma_m(s)$ and $\sigma_{m'}(s)$, while s is the spin- S momenta taking the values $-S, -S+1, \dots, S-1, S$, with $S = \frac{p^M-1}{2}$.

Performing the transformation given by Eqs. (7), (8) and (11) we obtain the Hamiltonian for the single-particle spin- S model

$$-\beta H_M(s) = \sum_{j=0}^{p^M} \sum_{j'=0}^{p^M} A_j(s) A_{j'}(s) F_{j, j'}, \quad (51)$$

where

$$F_{j, j'} = \sum_{\langle m, m' \rangle} \left[J_{m, m'} P_{j, m} P_{j', m'} + \frac{2h}{\gamma} (P_{j, m} + P_{j', m'}) \right], \quad (52)$$

the function $F_{j, j'}$ depends only of the lattice 'structure' and spin coupling parameter for a given system. Note that $F_{j, j'}$ does not depend of the variable s .

The Hamiltonian (51) can be understand as single particle Hamiltonian with large spin- S ($S = \frac{p^M-1}{2}$), similar to that discussed by Joseph [15]. Therefore the partition function simply becomes as the summation over spin- S momenta,

$$\mathcal{Z}_M = \sum_{s=-\frac{p^M-1}{2}}^{\frac{p^M-1}{2}} e^{-\beta H_M(s)}. \quad (53)$$

This is a different way to write the partition function of d-dimensional Ising model.

As a simple example, let us consider the spin-1/2 one-dimensional Ising model, with periodic boundary condition $\sigma_{M+1}(s) = \sigma_1(s)$ and assuming uniform coupling J between

spins, under longitudinal magnetic field h , so the Hamiltonian reads as

$$-\beta H_M(s) = \sum_{m=1}^M [J\sigma_m(s)\sigma_{m+1}(s) + h\sigma_m(s)], \quad (54)$$

then the function $F_{j,j'}$ for this particular case becomes

$$F_{j,j'} = \sum_{m=1}^M [JP_{j,m}P_{j',m+1} + \frac{h}{2}(P_{j,m} + P_{j',m+1})]. \quad (55)$$

Therefore, the partition function can be expressed by

$$\mathcal{Z}_M = \sum_{s=-\frac{2^M-1}{2}}^{\frac{2^M-1}{2}} \prod_{j=0}^{2^M-1} \prod_{j'=0}^{2^M-1} e^{A_j(s)A_{j'}(s)F_{j,j'}}. \quad (56)$$

Using the Eq. (56), we are able to write the partition function for fixed M , then the first few terms are given by

$$\mathcal{Z}_2 = e^{\frac{J}{2}+h} + 2e^{-\frac{J}{2}} + e^{\frac{J}{2}-h}, \quad (57)$$

$$\mathcal{Z}_3 = e^{\frac{3J}{4}+\frac{3h}{2}} + 3e^{-\frac{J}{4}+\frac{h}{2}} + 3e^{-\frac{J}{4}-\frac{h}{2}} + e^{\frac{3J}{4}-\frac{3h}{2}}, \quad (58)$$

$$\mathcal{Z}_4 = 4 + e^{J+2h} + 4e^h + 2e^{-J} + 4e^{-h} + e^{J-2h}, \quad (59)$$

$$\mathcal{Z}_5 = 5e^{\frac{J}{4}+\frac{h}{2}} + e^{\frac{5J}{4}-\frac{5h}{2}} + 5e^{-\frac{3J}{4}-\frac{h}{2}} + 5e^{-\frac{3J}{4}+\frac{h}{2}} + 5e^{\frac{J}{4}-\frac{h}{2}} + 5e^{\frac{J}{4}-\frac{3h}{2}} + e^{\frac{5J}{4}+\frac{5h}{2}} + 5e^{\frac{J}{4}+\frac{3h}{2}}. \quad (60)$$

In order to write the previous results in analogy to the solution obtained via transfer matrix. We define some basic quantities conveniently, such that the factors involving the largest (lowest) spin momenta are given by

$$a_M^\pm = \exp\left(-\frac{\beta}{M}H_M(\pm\frac{2^M-1}{2})\right) = e^{\frac{J}{4}\pm\frac{h}{2}}. \quad (61)$$

Similarly, we define also define the factor for next largest (lowest) spin momenta, which reads as

$$c_M^\pm = \exp\left(-\frac{\beta}{M}H_M(\pm\frac{2^M-3}{2})\right) = e^{\frac{M-4}{4M}J\pm\frac{M-2}{2M}h}, \quad (62)$$

the gap energy of spin- S between the largest (lowest) and next largest (lowest) energy could be defined as

$$\Delta E^\pm = -\beta \left(H_M(\pm\frac{2^M-1}{2}) - H_M(\pm\frac{2^M-3}{2}) \right) = J \pm h. \quad (63)$$

Let us define the following factor involving the energy gap of spin- S ,

$$b_M^\pm = \exp\left(\Delta E^\pm/4\right) = e^{\frac{J\pm h}{4}}. \quad (64)$$

Using the previous definition, and by using some algebraic manipulation, we can express the Eq. (56) alternatively as follow,

$$\mathcal{Z}_M = \left(\frac{a_M^+ + a_M^- + \sqrt{\left(a_M^+ - a_M^- \right)^2 + 4b_M^+ b_M^-}}{2} \right)^M + \left(\frac{a_M^+ + a_M^- - \sqrt{\left(a_M^+ - a_M^- \right)^2 + 4b_M^+ b_M^-}}{2} \right)^M. \quad (65)$$

Using the Eq. (65), we can verify the results obtained in (57) - (60), and higher order terms (not shown here). It is interesting to note that a_M^\pm and b_M^\pm is always independent of M .

Therefore the free energy in thermodynamic limit is given by

$$-\beta f = \lim_{M \rightarrow \infty} \ln(\mathcal{Z}_M) = \ln \left(e^{J/4} \cosh(h/2) + \sqrt{e^{J/2} \sinh(h/2) + e^{-J/2}} \right). \quad (66)$$

We have arrived to this results in some similar way as discussed by Joseph [15]. Note, that the Eq. (66) can be obtained using the transfer matrix method.

V. CONCLUSION

We have present the general spin-spin transformations between spin $S = \frac{2^M-1}{2}$ and a cluster of M spins $\sigma = \frac{p-1}{2}$ as well as the general inverse spin-spin transformation. We have discuss the application of our funding. In particular we have show one-to-one correspondence between a general spin-7/2 model on a d-dimensional lattice G and a three Ising model, each on the lattice G coupled by glue interactions. That results can be easily to extend to more general case, namely, that a general $d+1$ dimensional spin- $\frac{p-1}{2}$ model can be reduced to d -dimensional spin- S model with $S = \frac{p^M-1}{2}$. The representation of particles with the spin- S in terms of spins less than S seems to be very useful tool. We wish to clarify the critical properties of spin- S models with spin greater than 1/2 by using the present decomposition method in the future.

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VI. APPENDIX

Here we give the dependence of the coefficients $K_{a,b}, R_{a,b}, R_{a,b,c}$ and R from $J_{\alpha,\beta}$ and h_2, h_4, h_6

$$\begin{aligned} K_{1,1} = & J_{1,1} + \frac{61}{4} J_{1,3} + \frac{3481}{16} J_{1,5} + \frac{186901}{64} J_{1,7} + \frac{3721}{16} J_{3,3} + \frac{212341}{64} J_{3,5} \\ & + \frac{11400961}{256} J_{3,7} + \frac{12117361}{256} J_{5,5} + \frac{650602381}{1024} J_{5,7} + \frac{34931983801}{4096} J_{7,7} \end{aligned} \quad (67)$$

$$\begin{aligned} K_{1,2} = & 2 J_{1,1} + 29 J_{1,3} + \frac{2971}{8} J_{1,5} + \frac{37417}{8} J_{1,7} + \frac{3355}{8} J_{3,3} + \frac{42697}{8} J_{3,5} \\ & + \frac{8569045}{128} J_{3,7} + \frac{8566741}{128} J_{5,5} + \frac{212837399}{256} J_{5,7} + \frac{21014213935}{2048} J_{7,7} \end{aligned} \quad (68)$$

$$\begin{aligned} K_{1,3} = & 4 J_{1,1} + 46 J_{1,3} + \frac{2371}{4} J_{1,5} + 7606 J_{1,7} + \frac{1891}{4} J_{3,3} + 5776 J_{3,5} \\ & + \frac{4619941}{64} J_{3,7} + \frac{4389541}{64} J_{5,5} + \frac{108081833}{128} J_{5,7} + \frac{10558224391}{1024} J_{7,7} \end{aligned} \quad (69)$$

$$\begin{aligned} K_{2,2} = & 4 J_{1,1} + 55 J_{1,3} + \frac{2461}{4} J_{1,5} + \frac{112435}{16} J_{1,7} + \frac{3025}{4} J_{3,3} + \frac{135355}{16} J_{3,5} \\ & + \frac{6183925}{64} J_{3,7} + \frac{6056521}{64} J_{5,5} + \frac{276702535}{256} J_{5,7} + \frac{12641629225}{1024} J_{7,7} \end{aligned} \quad (70)$$

$$\begin{aligned} K_{2,3} = & 8 J_{1,1} + 86 J_{1,3} + \frac{1861}{2} J_{1,5} + \frac{84463}{8} J_{1,7} + \frac{1705}{2} J_{3,3} + \frac{72823}{8} J_{3,5} \\ & + \frac{3296245}{32} J_{3,7} + \frac{3103321}{32} J_{5,5} + \frac{140402443}{128} J_{5,7} + \frac{6351565585}{512} J_{7,7} \end{aligned} \quad (71)$$

$$\begin{aligned} K_{3,3} = & 16 J_{1,1} + 124 J_{1,3} + 1261 J_{1,5} + \frac{56491}{4} J_{1,7} + 961 J_{3,3} + \frac{39091}{4} J_{3,5} \\ & + \frac{1751221}{16} J_{3,7} + \frac{1590121}{16} J_{5,5} + \frac{71235151}{64} J_{5,7} + \frac{3191233081}{256} J_{7,7} \end{aligned} \quad (72)$$

$$\begin{aligned} K_{2,1} = & 21 J_{2,2} + \frac{3003}{8} J_{2,4} + \frac{41613}{8} J_{2,6} + \frac{41181}{8} J_{4,4} + \frac{8470293}{128} J_{4,6} \\ & + \frac{212094831}{256} J_{6,6} + 2 \gamma h_2 + 53 \gamma h_4 + \frac{6331}{8} \gamma h_6 \end{aligned} \quad (73)$$

$$\begin{aligned} K_{3,1} = & 42 J_{2,2} + \frac{1995}{4} J_{2,4} + \frac{25233}{4} J_{2,6} + \frac{22533}{4} J_{4,4} + \frac{4438005}{64} J_{4,6} \\ & + \frac{107571711}{128} J_{6,6} + 4 \gamma h_2 + 58 \gamma h_4 + \frac{3211}{4} \gamma h_6 \end{aligned} \quad (74)$$

$$\begin{aligned} K_{3,2} = & 84 J_{2,2} + \frac{1743}{2} J_{2,4} + 9624 J_{2,6} + \frac{17871}{2} J_{4,4} + \frac{3150213}{32} J_{4,6} \\ & + \frac{69380571}{64} J_{6,6} + 8 \gamma h_2 + 92 \gamma h_4 + \frac{2071}{2} \gamma h_6 \end{aligned} \quad (75)$$

$$R_{1,2} = 16 J_{2,2} + 424 J_{2,4} + 6331 J_{2,6} + 11236 J_{4,4} + \frac{335543}{2} J_{4,6} + \frac{40081561}{16} J_{6,6} \quad (76)$$

$$R_{1,3} = 64 J_{2,2} + 928 J_{2,4} + 12844 J_{2,6} + 13456 J_{4,4} + 186238 J_{4,6} + \frac{10310521}{4} J_{6,6} \quad (77)$$

$$R_{2,3} = 256 J_{2,2} + 2944 J_{2,4} + 33136 J_{2,6} + 33856 J_{4,4} + 381064 J_{4,6} + 4289041 J_{6,6} \quad (78)$$

$$R_{1,2,3} = 32 J_{2,2} + 656 J_{2,4} + 9542 J_{2,6} + 12296 J_{4,4} + 176891 J_{4,6} + \frac{20328841}{8} J_{6,6} \quad (79)$$

$$R_{2,1,3} = 64 J_{2,2} + 1216 J_{2,4} + 16804 J_{2,6} + 19504 J_{4,4} + 255376 J_{4,6} + \frac{13111501}{4} J_{6,6} \quad (80)$$

$$R_{3,1,2} = 128 J_{2,2} + 1664 J_{2,4} + 21128 J_{2,6} + 21344 J_{4,4} + 267824 J_{4,6} + \frac{6649981}{2} J_{6,6} \quad (81)$$

$$R_{1,3,2} = 24 J_{1,3} + 420 J_{1,5} + \frac{11613}{2} J_{1,7} + 732 J_{3,3} + \frac{23253}{2} J_{3,5} + 158637 J_{3,7} \\ + \frac{365505}{2} J_{5,5} + \frac{79674063}{32} J_{5,7} + \frac{2170481313}{64} J_{7,7} \quad (82)$$

$$R_{2,3,1} = 48 J_{1,3} + 840 J_{1,5} + 11613 J_{1,7} + 1320 J_{3,3} + 18933 J_{3,5} + 244005 J_{3,7} \\ + 258405 J_{5,5} + \frac{52190943}{16} J_{5,7} + \frac{1305707655}{32} J_{7,7} \quad (83)$$

$$R_{3,2,1} = 96 J_{1,3} + 1680 J_{1,5} + 23226 J_{1,7} + 1488 J_{3,3} + 20586 J_{3,5} + 264738 J_{3,7} \\ + 264810 J_{5,5} + \frac{26507103}{8} J_{5,7} + \frac{656029983}{16} J_{7,7} \quad (84)$$

$$R = 9 (256 J_{3,3} + 4480 J_{5,3} + 78400 J_{5,5} + 61936 J_{7,3} + 1083880 J_{7,5} + 14984641 J_{7,7}) \quad (85)$$

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